Structures for Interpolation, Decimation, and Nonuniform Sampling Based on Newton's Interpolation Formula

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Abstract:

The variable fractional-delay (FD) filter structure by Tassart and Depalle performs Lagrange interpolation in an efficient way. We point out that this structure directly corresponds to Newton's interpolation (backward difference) formula, hence we prefer to refer to it as the *Newton FD filter*. This structure does not function correctly when the fractional delay is made time-variant, e.g., in sample rate conversion. We present a simple modification that enables time-variant usage such as fractional sample rate conversion and nonuniform resampling. We refer to the new structure as the *Newton (interpolator) structure*. Almost all advantages of the Newton FD structure are preserved. Furthermore, we suggest that by transposing the Newton interpolator we obtain the *transposed Newton structure* which can be used in decimation as well as reconstruction of nonuniformly sampled signals, analogously to the transposed Farrow structure. The presented structures are a competitive alternative for the Farrow structure family when low complexity and flexibility are required.

1. Introduction

In [1][2][3], Tassart and Depalle as well as Candan derive an efficient implementation structure for FD filters, depicted in Fig. 1, from Lagrange's interpolation formula. It turns out that the obtained filter structure directly corresponds to Newton's (backward difference) interpolation formula [4] (with some subexpression sharing) which indeed is equivalent with Lagrange interpolation [5]. Newton's backward difference formula is

$$f(t+\tau) = \sum_{m=0}^{\infty} \frac{\tau^{(m)} \Delta^m f(t)}{m!}, \qquad (1)$$

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where

$$\tau^{(m)} = \prod_{k=0}^{m-1} (\tau + k) \qquad (2)$$

is the rising factorial, and Δ is the backward difference operator such that $\Delta^m f(t) = \Delta^{m-1} f(t) - \Delta^{m-1} f(t-1)$ and $\Delta^0 f(t) = f(t)$, resulting in

$$\Delta^m f(t) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(t-k). \qquad (3)$$

Newton's backward difference formula provides an efficient means to realise piecewise-polynomial interpolation for DSP. Its complexity is only $O(M)$ (where M is the interpolator order)–cf. equivalent Lagrange implementations based on the Farrow structure [6] having $O(M^2)$ complexity [3]. The subfilters are multiplier-free and extremely simple. The structure is modular, as highlighted by the grey shading in Fig. 1, and the interpolator order can be changed in real time [3].

Unfortunately, the structure presented in Fig. 1 does not function correctly in sample rate conversion (SRC). Because the multiplications are performed between the subfilters, making them time-variant will result in incorrect output. This is because each output sample should only depend on the current value of the delay parameter D; in Fig. 1, past values of D contribute to the output through the delayed paths through the subfilters. Therefore, the structure in Fig. 1 is only useful in single-rate, time-invariant or slowly-varying fractional-delay filtering.

We propose a slightly modified structure that allows arbitrary resampling, including increasing the sample rate by arbitrary, also fractional, factors (fractional interpolation). We also point out that the structure can be transposed to obtain a decimator structure that possesses all the advantages of the Newton interpolation structure.

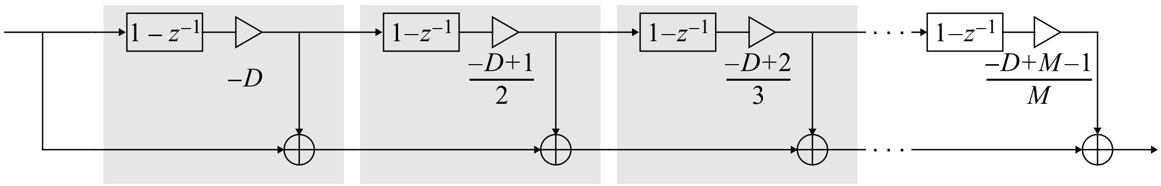


Figure 1. The fractional-delay filter structure proposed in [1][3], based on Newton's interpolation formula.

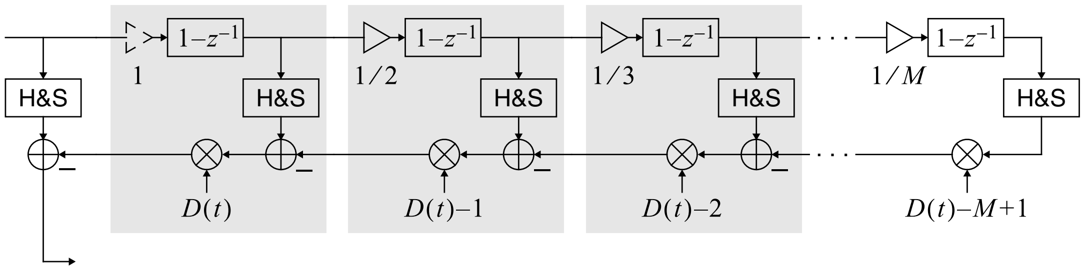


Figure 2. The Newton interpolator structure suitable for sample rate conversion. The hold & sample (H&S) blocks perform the sampling at the output sample instants.

2. The Newton structure for interpolation

In order to allow fractional SRC and arbitrary resampling, the Newton structure must work correctly with a time-variant fractional delay. This is achieved through two simple steps: (*i*) We invert the summation order at the output part of the structure from that presented in [1][3] (this was already done in [2]). (*ii*) The time-varying multiplications can now be implemented in the high-rate part between the adders. The improved structure is shown in Fig. 2. We refer to it as the *Newton interpolator structure* or the Newton structure for short. Also the improved structure is modular, permitting changing the interpolator order in real time. In single-rate FD filtering, the improved structure is equivalent to [1][2][3].

In Fig. 2, the H&S blocks stand for hold & sample, i.e., each output sample obtains the value of the previously arrived input sample.

In fractional interpolation, i.e., increasing the sample rate by a fractional factor, we use the common notation illustrated in Fig. 3. The time interval between the previous input sample and the next output sample to be generated is expressed using the *fractional interval* variable μ which is normalised with respect to the input sample interval so that $\mu \in [0, 1)$.

Interpolation of uniformly spaced input samples can be modelled as convolution [5], leading to the generic model depicted in Fig. 4 [7]. The continuous-time (CT) linear time-invariant (LTI) model filter is piecewise polynomial, with $M + 1$ pieces, each with duration equal to the input sample interval T_{in}. Hence the impulse response length is $(M + 1)T_{\text{in}}$.

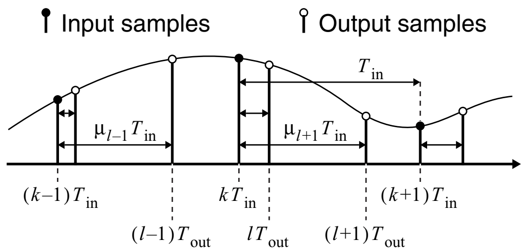
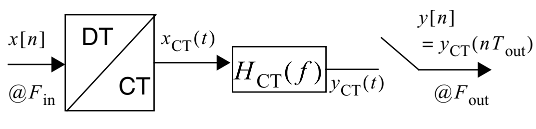


Figure 3. Definition of the fractional interval μ for interpolation.



$$x_{\text{CT}}(t) = \frac{1}{F_{\text{in}}} \sum_n x[n]\delta\left(t - \frac{n}{F_{\text{in}}}\right)$$

Figure 4. The generic model for SRC by arbitrary factors.

The composite transfer function of m cascaded subfilters is

$$(1 - z^{-1})^m = \sum_{n=0}^{m} (-1)^n \binom{m}{n} z^{-n}, \tag{4}$$

cf. (3). The output of the interpolator is

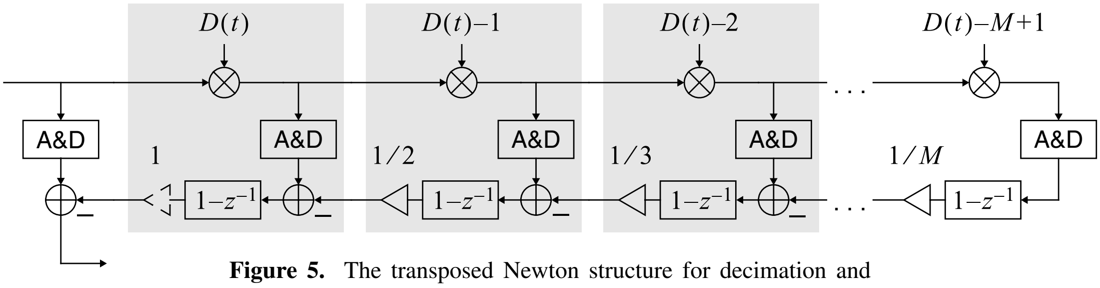


Figure 5. The transposed Newton structure for decimation and reconstruction of signals from nonuniformly spaced samples.

$$y((k+\mu)T_{\text{in}}) = \sum_{n=0}^{M} h((n+\mu)T_{\text{in}})x[k-n]$$

$$= \sum_{n=0}^{M} x[k-n](-1)^n \sum_{m=n}^{M} (-1)^m \binom{m}{n} \frac{(D_0-\mu)_m}{m!} \quad (2.1)$$

where

$$\binom{m}{n} = 0, \quad n < 0 \vee n > m \quad (5)$$

for $m \geq 0$, and

$$(x)_m = \prod_{k=0}^{m-1} (x-k) \quad (6)$$

is the falling factorial. The delay of the interpolator is $D_0 T_{\text{in}}$. The parameter D_0 can be chosen quite freely, but the best amplitude response and linear phase response are obtained with $D_0 = (M+1)/2$ [1].

The continuous-time model impulse response of the interpolator is then (cf. the expression of the filter input in Fig. 4)

$$h((n+\mu)T_{\text{in}}) = \frac{1}{T_{\text{in}}} \sum_{m=n}^{M} (-1)^{n+m} \binom{m}{n} \frac{(D_0-\mu)_m}{m!}. \quad (7)$$

The reversed summation order in the high-rate part comes with a price: the structure is more costly to pipeline than those in [1][3] because the signal paths cannot share pipeline registers.

3. The transposed Newton structure

There exists a duality[1] between decimation and interpolation that allows transforming a decimator into an interpolator and vice versa through network transposition [7]. By transposing the Newton interpolator, we obtain the structure depicted in Fig. 5. We refer to this as the *transposed Newton structure*. The transpose is obtained by inverting the flow direction of all signals and replacing each block with its dual. For instance, the H&S block is replaced with the accumulate & dump

1. There exist a number of definitions for duality, including the adjoint. Here we use the generalised duality/transpose as defined in [7].

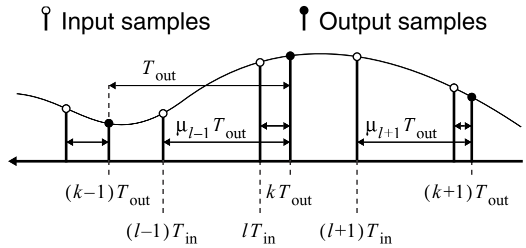


Figure 6. Definition of the fractional interval μ for the transposed structure (dual of interpolation).

(A&D) block, which sums up all its input samples since the previous output sample. This is also the most straightforward way to obtain the transposed Farrow structure from the Farrow structure[2] [9].

The output samples of the transposed Newton structure are uniformly spaced, but the input samples may arrive at arbitrary time instants. The generic SRC model (Fig. 4) is valid also for the transposed Newton structure. The model impulse response is again piecewise-polynomial, now with the piece duration equal to the *output* sample interval. The model impulse response is obtained by replacing T_{in} with T_{out} in (7) and redefining μ according to Fig. 6 (reflecting the duality between decimation and interpolation). For an input sample arriving at time instant t , the fractional interval is

$$\mu(t) = \left\lceil \frac{t}{T_{\text{out}}} \right\rceil - \frac{t}{T_{\text{out}}} \in [0, 1). \quad (8)$$

For fractional decimation, the fractional interval for the l^{th} input sample is

$$\mu_l = \left\lceil \frac{lT_{\text{in}}}{T_{\text{out}}} \right\rceil - \frac{lT_{\text{in}}}{T_{\text{out}}}. \quad (9)$$

The impulse response in the generic model is now

$$h((n+\mu)T_{\text{out}}) = \frac{1}{T_{\text{out}}} \sum_{m=n}^{M} (-1)^{n+m} \binom{m}{n} \frac{(D_0-\mu)_m}{m!} \quad (10)$$

2. The structure in [8] (transposed structure I in [9]) is not the true transpose of the Farrow structure even though the duality of responses holds.

with integer n. Again, $D_0 = (M+1)/2$ for the best response. In the frequency response, the model filter has $M+1$ zeros at each (nonzero) integer multiple of the output sample rate, hence realising antialiasing regardless of the decimation factor.

The transposed Newton structure is able to receive input samples at arbitrary time instants, which makes it a potential building block for reconstruction of signals from nonuniformly spaced samples (e.g., in algorithms like [10][11]), as earlier suggested for the transposed Farrow structure in [12].

The transposed Newton structure shares the advantages and disadvantages of the Newton interpolator, such as modularity, $O(M)$ complexity and the inefficient zero locations.

4. Computational complexity

In interpolation by factor R, the Newton structure will perform $(1+R)M$ additions and $(1+R)M$ multiplications per input sample on average. In decimation by R, the transposed Newton structure will perform $(R-1)(1+M)+2M$ additions and $(1+R)M$ multiplications per output sample. The first term in the addition count comes from the A&D block. Multiplication by a constant inverse of a small integer requires only few additions/subtractions.

Unambiguous complexity comparison between the proposed structures and alternatives, mainly the Farrow family, would require specifying the implementation technology and the SRC factor. However, the following points can be made: (*i*) The basis multipliers are more complex in the Newton structures (integer part present in the time-variant coefficients) than in Farrow structures (no integer part). Hence, large SRC factors are unfavourable to the Newton family. (*ii*) If the Lagrange response suffices, the ultimate simplicity of the subfilters makes the Newton family superior to the Farrow structure when the SRC factor is small. (*iii*) The response of the Newton structures can be improved only by increasing the order (i.e., number of stages). In designs with a low oversampling factor and/or strict performance requirements, this may lead to a very high filter order. In such cases, an optimised Farrow design with a non-Lagrange response will have a lower complexity and smaller delay.

5. Conclusions

The proposed structures allow efficient piecewise Newton interpolation for SRC and arbitrary resampling as well as its dual for decimation and reconstruction of nonuniformly sampled signals. The advantages of the proposed structures include low, $O(M)$ complexity (high orders are feasible at the cost of a long delay), very simple subfilters and run-time adjustability of the filter order. As a drawback, the basis multipliers running at the high-rate end of the filter have longer wordlengths than in the Farrow counterparts.

Due to their simplicity, the Newton structures may be useful as building blocks of more complicated algorithms for interpolation, decimation, and reconstruction of nonuniformly sampled signals.

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